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Asymptotic expansions of the Biot-Savart law for a slender vortex with core variation

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Abstract. The Method of Matched Asymptotic Expansion of Singular Integrals (MAESI) is used to expand the Biot-Savart law in terms of different parameters. This method is first used to find, in terms of the small distance r to a line vortex, the first orders of the known expansion of the potential flow induced by this line vortex. This method is also used to easily compare two equations of motion of a slender vortex filament: the one obtained in an ad-hoc way by a cut-off line-integral technique and the other derived from the Navier-Stokes equations by Callegari and Ting. Finally, this method is used to give the inner expansion of the flow induced by a slender vortex in terms of its slenderness ϵ. This is the first inner expansion up to order one in terms of ϵ of the Biot-Savart law for a slender vortex. An application of this inner expansion is then given to find the induced velocity of a family of non-circular vortex rings with axisymmetric axial-core variation. In order to understand the time-evolution of these initial conditions to the Navier-Stokes equations, a short time scale is introduced. A quasi-hyperbolic system that describes the leading-order dynamics of the axisymmetric axial core variation on a curved slender vortex filament is finally extracted from the Navier-Stokes equations.

Key words: Biot-Savart law, matched asymptotic expansions, singular integrals, vortex filament, axial core variation.

1. Introduction

An integral with a small parameter may become singular if this parameter is set to zero. The method of Matched Asymptotic Expansion of Singular Integrals (MAESI) [1, pp. 98–104], [2, pp. 341–349] is a well established method used to obtain the expansion of such integrals in terms of a small parameter. In an incompressible inviscid or viscous fluid the Biot-Savart law is an integral equation that relates the velocity field to that for the vorticity. In analytical studies of slender vortex filaments this integral often becomes singular in terms of some small parameter. For example, the velocity $\mathbf{v}$ induced by a line vortex $\mathcal{C}$ of circulation Γ is

$$\mathbf{v}(\mathbf{x}) = \frac{\Gamma}{4\pi}\int_{\mathcal{C}} \frac{\mathbf{t}(a') \times \big(\mathbf{x} - \mathbf{X}(a')\big)}{|\mathbf{x} - \mathbf{X}(a')|^3}\mathrm{d}a', \tag{1}$$

where $\times$ is the cross-product, $\mathbf{t}$ is a tangent vector to the line and $\mathbf{X} = \mathbf{X}(a)$ is a function which denotes a point on this curve as a function of arclength a. This expression of $\mathbf{v}$ is a singular integral in terms of the distance r to the line. Its expansion in terms of r is given by several authors [3], [4, pp. 33–38] in the form

$$\mathbf{v}(r) = \frac{\Gamma}{2\pi r}\mathbf{e}_\varphi + \frac{\Gamma}{4\pi}K \log\frac{\mathcal{L}}{r}\mathbf{b} + \mathbf{Q}_f + O(r), \tag{2}$$

where K is the local curvature of the line, $\mathbf{b}$ the binormal vector and $\mathbf{e}_\varphi$ the orthoradial vector in a normal plan to the filament (Figure 1). The length $\mathcal{L}$ and the finite part $\mathbf{Q}_f$ of the self-induced velocity are not often given, but can be found in Callegari and Ting [5].

In this paper we use the MAESI method to expand the Biot-Savart law in terms of different parameters. This method is first used in Section 2 to find, in terms of the small distance r to a line vortex, the first orders of the expansion of the potential flow (1) induced by this line. With this method the derivation of this known expansion becomes straightforward. So this gives a new interesting derivation of this expansion and is an alternative to the technique of the osculating circle initially used by Widnall *et al.* [6] and used by Moore and Saffman [7, 4]. As this method has rarely been used in the field of vortex dynamics and will be used in a more complex situation in Section 4, its successive steps are fully given in Section 2 to show how it works. The expansion is obtained up to order $O(r)$ by Fukumoto and Miyazaki [8]. For the first time all global integral parts are explicitly given.

This method is also used in Section 3 to easily compare two equations of motion of a slender vortex filament, namely the one obtained in an ad-hoc way by a cut-off line integral technique [9, 10] and the other derived from the Navier-Stokes equations by Callegari and Ting [5]. This comparison gives the cut-off length as a function of the inner structure parameters C_v and C_w defined by Callegari and Ting.

Finally, the method is used in Section 4 to give the inner expansion of the flow induced by a slender vortex in terms of its slenderness ϵ. This is the first inner expansion up to $O(1)$ in terms of ϵ of the Biot-Savart law for a slender vortex. The successive steps of this more complex use of the MAESI method are not given, because this method has previously been described in Section 2 for the line vortex.

An application of this inner expansion of the Biot-Savart law is then given in Section 5 to find the induced velocity of a family of non-circular vortex rings with axisymmetric axial-core variation. These vortex rings with core variation are interesting initial conditions to the Navier-Stokes equations.

Finally, in order to understand the time-evolution of these initial conditions, a short time scale is introduced in Section 6. This time is in-between the time of the evolution of a non-axisymmetric core and the time of motion of a curved vortex. In this Section 6 a quasi-hyperbolic system, describing the leading-order dynamics of axisymmetric axial core variation on curved slender vortex filament, is finally extracted from the Navier-Stokes equations. This is of interest when compared to systems obtained in an ad-hoc way such as the one proposed by Lundgren and Ashurst [11].

2. The potential flow induced by a line vortex near this line

The closed line vortex $\mathcal{C}$ of circulation Γ and length S is described parametrically by use of a function $\mathbf{X} = \mathbf{X}(s)$ which denotes a point on the curve as a function of the parameter s with $s \in [-\pi, \pi[$. At each point of this curve the Frenet vector basis $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ exists with the tangent, normal, and binormal vectors (Figure 1), respectively. Here and throughout this paper the differentiation $\partial f/\partial x$ of a function f with respect to its variable x is denoted by f_x. The variable $\sigma(s) = |\mathbf{X}_s|$ is introduced and is equal to 1 if s is an arclength denoted by a. As we are interested in finding the velocity field near the line $\mathcal{C}$, we introduce a *local curvilinear coordinate system* $\mathbf{M}(r, \varphi, s)$ and the curvilinear vector basis $(\mathbf{e}_r, \mathbf{e}_\varphi, \mathbf{t})$ valid near this line. This system is defined in the following manner: if $\mathbf{P}(s)$ is the projection on $\mathcal{C}$ of a point $\mathbf{M}$

near the curve, then **PM** is in the plane $(\mathbf{n}, \mathbf{b})$ and thus polar coordinates (r, φ) can be used in this plane with the associated polar vectors $(\mathbf{e}_r, \mathbf{e}_\varphi)$. The induced velocity is then given by

$$\mathbf{v}(r, \varphi, a) = \frac{1}{4\pi} \int_{\mathcal{C}} \frac{\mathbf{t}(a') \times \left(\mathbf{x} - \mathbf{X}(a')\right)}{|\mathbf{x} - \mathbf{X}(a')|^3} \mathrm{d}a', \tag{3}$$

where $\mathbf{x} = \mathbf{X}(a) + r\mathbf{e}_r(\varphi, a)$. Here all lengths are non-dimensionalized by a characteristic length L of the same order as the inverse of the local curvature K and the velocity field by Γ/L. The expansion of $\mathbf{v}$ in terms of r is of interest because it is used in the asymptotic derivation of the leading-order equation of motion of a slender vortex filament [5] in order to perform the asymptotic matching between the outer region of the slender filament and the inner region of the core.

This expansion of $\mathbf{v}$ in terms of r was first derived with the technique of the osculating circle by Widnall *et al.* [6] and then this technique was also used by Moore and Saffman [7, 4]. The MAESI method is straightforward and gives a new interesting derivation. In order to show how it works, and as this method has rarely been used in the field of vortex dynamics, we found it interesting to quickly describe its successive steps in this section for readers who may not know of this method. It is also easier to give a description of this method in this simple case of a line vortex than in the more advanced case of a slender vortex filament as we use it in Section 4. So this gives a useful introduction to this following section. As it will be fully described in the following, this method consists in splitting the integral into two parts. In an *outer region* outside a neighbourhood of the point $\mathbf{P}(a)$ the integrand is expanded in terms of r with $a^* = a' - a$ held fixed and then integrated. In an *inner region* in a neighbourhood of $\mathbf{P}(a)$ the stretched inner variable $\bar{a} = a^*/r$ is introduced. The obtained integrand is expanded in terms of r with the stretched inner variable held fixed and then integrated. The last step is the asymptotic matching which consists in adding these two integrated expansions.

First, in order to move the singularity from a to 0, the change of variable $a^* = a' - a$ is performed and the integral (3) becomes

$$\mathbf{v}(r, \varphi, a) = \frac{1}{4\pi} \int_{-S/2}^{+S/2} \mathbf{K}(r, \varphi, a, a^*) \mathrm{d}a^*, \tag{4}$$

where $\mathbf{K}(r, \varphi, a, a^*) = \mathbf{t}(a + a^*) \times (\mathbf{x} - \mathbf{X}(a + a^*)) / |\mathbf{x} - \mathbf{X}(a + a^*)|^3$. Following the MAESI method, the small intermediate parameter η, such that $r \ll \eta \ll 1$, is then introduced and the integral is split into two parts $\mathbf{v}(r, \varphi, a) = \mathbf{Out} + \mathbf{In}$, where

$$\mathbf{Out} = \frac{1}{4\pi} \int_{-S/2}^{-\eta} \mathbf{K} \mathrm{d}a^* + \frac{1}{4\pi} \int_{\eta}^{+S/2} \mathbf{K} \mathrm{d}a^* \quad \text{and} \quad \mathbf{In} = \frac{1}{4\pi} \int_{-\eta}^{\eta} \mathbf{K} \mathrm{d}a^*. \tag{5}$$

The stretched variable $\bar{a} = a^*/r$ is then introduced in the inner region and gives

$$\mathbf{In} = \frac{1}{4\pi} r \int_{-\eta/r}^{\eta/r} \tilde{\mathbf{K}} \mathrm{d}\bar{a},$$

where $\tilde{\mathbf{K}}(r, \varphi, a, \bar{a}) = \mathbf{K}(r, \varphi, a, r\bar{a})$ has been defined.

In a first step we perform the *outer expansion* by finding the expansion in terms of r of the outer part **Out**. In order to do so, the expansion of $\mathbf{K}$ in terms of r is first found to be

$$\begin{aligned} \mathbf{K} = {} & \mathbf{t}(a + a^*) \times \mathbf{d} / |\mathbf{d}|^3 - r\mathbf{e}_r \times \mathbf{t}(a + a^*) / |\mathbf{d}|^3 \\ & -3r \left(\mathbf{e}_r(\varphi, a) \cdot \mathbf{d} / |\mathbf{d}|^5\right) \mathbf{t}(a + a^*) \times \mathbf{d} + O(r^2), \end{aligned} \tag{6}$$

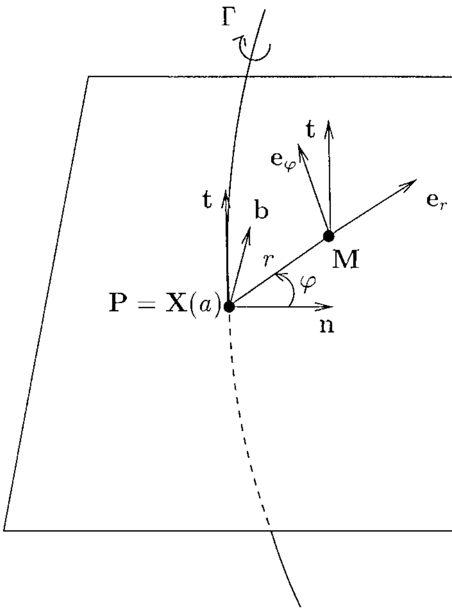

Figure 1. The centerline and the local co-ordinates of the vortex ring.

where $\mathbf{d} = \mathbf{X}(a) - \mathbf{X}(a + a^*)$. This expansion (6) is then integrated with respect to a^* which gives the sought outer expansion.

In a second step we perform the *inner expansion* by finding the expansion in terms of r of the inner part **In**. In order to do so, the expansion of $\tilde{\mathbf{K}}$ in terms of r is first found to be

$$\begin{aligned} r\tilde{\mathbf{K}} = {} & \frac{\mathbf{e}_\varphi}{rg^3} + \frac{3K\bar{a}^2\cos\varphi\mathbf{e}_\varphi}{2g^5} + \frac{K\bar{a}^2\mathbf{b}}{2g^3} + r\left(\frac{\bar{a}^4K^2\mathbf{e}_\varphi}{8g^5} + \frac{15K^2\bar{a}^4\cos^2\varphi\mathbf{e}_\varphi}{8g^7}\right) \\ & + r\left(\frac{3\bar{a}^4K^2\cos\varphi\mathbf{b}}{4g^5} - \frac{\bar{a}^2\left(KT\cos\varphi - K_a\sin\varphi\right)\mathbf{t} + \bar{a}^2K^2\mathbf{e}_\varphi}{2g^3}\right) + O\left(\frac{r^2}{g^3}\right), \end{aligned} \tag{7}$$

where $g = \sqrt{1 + \bar{a}^2}$. Then this expansion (7) is integrated with respect to $\bar{a}$. All these integrals have analytical expressions which are easily found and this gives the sought inner expansion. There are no longer integrals in this expansion.

In a last step we perform the *asymptotic matching* by expanding the outer expansion in terms of $\eta \ll 1$, by expanding the inner expansion in terms of $\eta/r \gg 1$ and by adding these two expansions. This proceeds as follows. In order to expand the outer expansion in terms of $\eta \ll 1$ and to remove its singularity in terms of η, the singular behaviour of each integrand near $a^* = 0$ of the integrals in this outer expansion are studied. For example the singular behaviour near $a^* = 0$ of the first term in Equation (6) is

$$\mathbf{t}(a + a^*) \times \mathbf{d}/|\mathbf{d}|^3 = K(a)\mathbf{b}(a)/2\left|a^*\right| + O(1),$$

and so we can write

$$\begin{aligned} \int_\eta^{+S/2} \frac{\mathbf{t}(a + a^*) \times \mathbf{d}}{|\mathbf{d}|^3}\mathrm{d}a^* = {} & \int_\eta^{+S/2} \left[\frac{\mathbf{t}(a + a^*) \times \mathbf{d}}{|\mathbf{d}|^3} - \frac{K(a)\mathbf{b}(a)}{2|a^*|}\right]\mathrm{d}a^* \\ & + \int_\eta^{+S/2} \frac{K(a)\mathbf{b}(a)}{2|a^*|}\mathrm{d}a^*. \end{aligned} \tag{8}$$

As the singular behaviour is now removed in the first integral at the right-hand side of Equation (8), its expansion in terms of η is then simply found by means of a Taylor expansion and the second integral is easily integrated. When this is done for each term, we have

$$\begin{aligned}\mathbf{Out} = \;& \mathbf{A}(a) + r(\mathbf{B}(\varphi, a) - 3\mathbf{C}(\varphi, a)) - \frac{1}{4\pi}K(a)\log\frac{2\eta}{S}\mathbf{b}(a)\\ &+\frac{r}{4\pi}\left(\frac{1}{\eta^2} - \frac{4}{S^2} + \frac{3}{4}K^2\log\frac{2\eta}{S}\right)\mathbf{e}_\varphi\\ &+\frac{r}{4\pi}(-\sin\varphi K_a + \cos\varphi KT)\log\frac{2\eta}{S}\mathbf{t} - \frac{3}{8\pi}rK^2\cos\varphi\log\frac{2\eta}{S}\mathbf{b}\\ &+O(\eta^2) + O(\eta r) + O(r^2),\end{aligned} \tag{9}$$

where T is the local torsion of the line vortex and $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are given by

$$\mathbf{A}(a) = \frac{1}{4\pi}\int_{-S/2}^{+S/2}[\frac{\mathbf{t}(a+a^*)\times\mathbf{d}}{|\mathbf{d}|^3} - \frac{K(a)\mathbf{b}(a)}{2|a^*|}]\mathrm{d}a^*, \tag{10}$$

$$\mathbf{B}(\varphi, a) = \frac{1}{4\pi}\mathbf{e}_r(\varphi, a)\times\int_{-S/2}^{+S/2}[-\frac{\mathbf{t}(a+a^*)}{|\mathbf{d}|^3} - f^{\mathrm{B}}(a, a^*)]\mathrm{d}a^*, \tag{11}$$

$$\mathbf{C}(\varphi, a) = \frac{1}{4\pi}\int_{-S/2}^{+S/2}[\frac{\mathbf{e}_r(\varphi, a)\cdot\mathbf{d}}{|\mathbf{d}|^5}[\mathbf{t}(a+a^*)\times\mathbf{d}] - f^{\mathrm{C}}(a, a^*)]\mathrm{d}a^*, \tag{12}$$

with

$$\begin{aligned} f^{\mathrm{B}}(a, a^*) &= -\frac{1}{|a^*|^3}[\mathbf{t}(a) + K(a)\mathbf{n}(a)a^* + \mathbf{k}\frac{a^{*2}}{2}],\\ \mathbf{k} &= K_a(a)\mathbf{n}(a) + K(a)T(a)\mathbf{b}(a) - \frac{3}{4}K^2(a)\mathbf{t}(a),\\ f^{\mathrm{C}}(a, a^*) &= -\frac{K^2(a)\mathbf{b}(a)\cos(\varphi)}{4|a^*|}.\end{aligned}$$

The inner expansion is then easily expanded in terms of $\eta/r \gg 1$ and gives

$$\begin{aligned} 4\pi\,\mathbf{In} = \;& \frac{2\mathbf{e}_\varphi}{r} - \frac{r\mathbf{e}_\varphi}{\eta^2} + \frac{3}{4}\frac{r^3}{\eta^4}\mathbf{e}_\varphi + K\cos\varphi\left[1 - \frac{3}{2}\left(\frac{r}{\eta}\right)^2\right]\mathbf{e}_\varphi\\ &+K\left[-1 + \log 2 + \log\frac{\eta}{r} + \frac{3}{4}\left(\frac{r}{\eta}\right)^2\right]\mathbf{b}(a)\\ &-2r\left[-\frac{K^2}{8}\mathbf{e}_\varphi - \frac{3}{4}K^2\cos\varphi\mathbf{b}(a)\right]\left(-\frac{4}{3} + \log 2 + \log\frac{\eta}{r}\right)\\ &-r\left[KT\cos\varphi - K_a\sin\varphi\right]\left(-1 + \log\frac{\eta}{r} + \log 2\right)\mathbf{t}(a)\\ &-rK^2\left(-1 + \log\frac{\eta}{r} + \log 2 - \frac{3}{4}\cos^2\varphi\right)\mathbf{e}_\varphi\\ &+O(r^2) + O\left(\frac{r^5}{\eta^6}\right) + O\left(\frac{r^4}{\eta^4}\right).\end{aligned} \tag{13}$$

Finally, the expansions (9) and (13) are added and $\mathbf{b}(a) = \sin(\varphi)\mathbf{e}_r + \cos(\varphi)\mathbf{e}_\varphi$ is replaced in the order $O(r)$. As expected, the intermediate parameter η disappears and we end up with the following expansion of the velocity $\mathbf{v}$ near the line vortex

$$\begin{aligned}\mathbf{v}(r \to 0, \varphi, a) = & \frac{1}{2\pi r}\mathbf{e}_\varphi + \frac{K}{4\pi}\cos\varphi\mathbf{e}_\varphi + \mathbf{A} + \frac{K}{4\pi}\left[\log\frac{S}{r} - 1\right]\mathbf{b} \\ & + r\mathbf{I} + \left(\mathbf{B} - 3\mathbf{C} - \frac{1}{\pi S^2}\mathbf{e}_\varphi\right) r + O(r^2 \log r),\end{aligned} \tag{14}$$

where

$$\begin{aligned}\mathbf{I} = & \frac{3}{16}\frac{K^2}{\pi}\left[\left(\mathbf{e}_r \sin 2\varphi + \mathbf{e}_\varphi \cos 2\varphi\right)\left(\log\frac{S}{r} - \frac{4}{3}\right)\right] + \frac{3}{16}\frac{K^2}{\pi}\left[\frac{1}{2}\mathbf{e}_\varphi \cos 2\varphi + \frac{1}{18}\mathbf{e}_\varphi\right] \\ & + \frac{1}{4\pi}\left(K_a \sin\varphi - KT\cos\varphi\right)\left[\log\frac{S}{r} - 1\right]\mathbf{t}.\end{aligned}$$

Fukumoto and Miyazaki [8] have already given this expansion (14), but it is for the first time that the expressions of the terms $\mathbf{B}$ and $\mathbf{C}$ are given. A comparison between (14) and (2) shows that $\mathcal{L} = S$ and that the exact value of $\mathbf{Q}_f$ is

$$\mathbf{Q}_f = \frac{\Gamma K}{4\pi}\left(\cos\varphi\mathbf{e}_\varphi - \mathbf{b}\right) + \Gamma\mathbf{A}.$$

This expansion (14) can also be used to easily obtain the induced velocity of a infinite *non-closed* line vortex. In order to do so, we consider that this line is composed of a central part of length S around the point where the velocity is sought and two semi-infinite parts on both sides of this central part to complete this line. We easily obtain the expansion of the velocity near this open line by applying (14) to the central part of this line and by adding the induced velocity of the two semi-infinite parts.

3. Comparison between the Callegari and Ting equation of motion and a cut-off technique

In this section we derive a relation between the cut-off-length parameter that appears in the cut-off line-integral technique [9, 10] and the inner structure parameters C_v and C_w defined by Callegari and Ting [5]. Such kind of comparison between an asymptotic equation of motion and a cut-off technique was first performed by Widnall *et al.* [6, 12] and then by Moore and Saffman [7, 4]. This was left to be done with the Callegari and Ting equation of motion.

Let us recall that a *slender vortex ring* of circulation Γ is a field of vorticity which is non-zero only in the neighbourhood of a three-dimensional curve $\mathcal{C}$, called the *centerline*. This curve is described parametrically by a function $\mathbf{X} = \mathbf{X}(s, t)$ which denotes a point on the curve as a function of the parameter s, with $s \in [-\pi, \pi[$, and the time t. The thickness δ of the ring is of order l and the other length scales, for example the local *curvature* K or the length S of $\mathcal{C}$, are of the same order L. Since the vortex is slender, a small parameter $\epsilon \ll 1$ is defined as the ratio l/L.

Using a careful matched asymptotic expansion in the Navier-Stokes equations, Callegari and Ting [5] found the following equation of motion

$$\partial\mathbf{X}/\partial t = \mathbf{Q} + \frac{K(s,t)}{4\pi}\left[-\log\epsilon + \log(S) - 1 + C_v(t) + C_w(t)\right]\mathbf{b}(s,t), \tag{15}$$

where $\mathbf{Q} = \mathbf{A}(s,t) - [\mathbf{A}(s,t)\cdot\mathbf{t}(s,t)]\,\mathbf{t}(s,t)$ with

$$\mathbf{A}(s,t) = \frac{1}{4\pi}\int_{-\pi}^{+\pi} \sigma(s+s',t)\left[\frac{\mathbf{t}(s+s',t)\times(\mathbf{X}(s,t)-\mathbf{X}(s+s',t))}{|\mathbf{X}(s,t)-\mathbf{X}(s+s',t)|^3} - \frac{K(s,t)\mathbf{b}(s,t)}{2\,|\lambda(s,s',t)|}\right]\mathrm{d}s',$$

and $\lambda(s,s',t) = \int_{s}^{s+s'} \sigma(s^*,t)\mathrm{d}s^*$. Here the velocity field is non-dimensionalized by Γ/L and all lengths by L. In this Equation (15), $C_v(t)$ and $C_w(t)$ are known functions [5] which describe the orthoradial and axial evolution of the inner velocity in the core. Equation (15) holds for a vortex ring with axisymmetric structure at leading order and no axial core variation at this order.

Prior to this asymptotic derivation of the equation of motion or the ones of Widnall *et al.* [6], and Moore and Saffman [7], the logarithmic singularity in terms of r which appears in (14) had been avoided by ad-hoc de-singularization techniques. For example, with the cut-off integral technique [9, 10] an ad-hoc cut-off of the line integral (3) gives a de-singularization of this integral in terms of the distance r to this line and yields the equation of motion:

$$\partial\mathbf{X}/\partial t = \frac{1}{4\pi}\int_I \sigma(s',t)\frac{\mathbf{t}(s',t)\times\big(\mathbf{X}(s,t)-\mathbf{X}(s',t)\big)}{|\mathbf{X}(s,t)-\mathbf{X}(s',t)|^3}\mathrm{d}s', \tag{16}$$

where $I = [0,2\pi[\backslash[s-s_c, s+s_c[$ and s_c is an unknown small parameter called the *cut-off length*.

This integral (16) is *singular* in terms of the small parameter s_c and can be expanded in terms of this parameter. In fact, the integral in Equation (16) is the same as the integral in Equation (5) of **Out**, if η is put equal to s_c. So one simply obtains this expansion of (16) by replacing η by s_c in (9). The comparison between this expansion and (15) leads to

$$s_c(s,t) = \frac{\epsilon}{2\sigma(s,t)}\exp\left(1 - C_v(t) - C_w(t)\right). \tag{17}$$

This gives the relation between the cut-off length s_c, the reduced thickness ϵ and the inner-core parameters $C_v(t)$ and $C_w(t)$ of Callegari and Ting. So Equations (16–17) are equivalent to Equation (15), except that, when s_c of (17) is plugged into (16), the integral is singular in terms of ϵ, while the integral $\mathbf{A}$ in (15) is not. This comparison can also be performed for other kinds of ad-hoc de-singularization [13]. The de-singularized integrals subjected to these ad-hoc techniques are still singular integrals in terms of their ad-hoc parameter of de-singularization.

4. The inner expansion of the velocity field induced by a slender vortex

In this section the inner expansion of the flow induced by a slender vortex in terms of its slenderness ϵ is carried out. This is the first inner expansion up to $O(1)$ in terms of ϵ of the Biot-Savart law for a slender vortex. In order to perform this expansion, we use the MAESI method that was previously described in Section 2 for the simpler case of a curved line vortex. The outer expansion up to $O(1)$ in terms of ϵ of the Biot-Savart law for this slender vortex is also proved to be the velocity field (3) induced by a line vortex.

As defined in the previous section, a *slender vortex ring* is a solenoidal field of vorticity $\boldsymbol{\omega}(\mathbf{x})$ which is non-zero only in the neighbourhood of a three-dimensional curve $\mathcal{C}$. The flux

Γ of vorticity in each section of the ring is a constant and the vortex ring may have an axial velocity flux of strength m. One can distinguish an *outer problem* defined by the *outer limit* $\epsilon \to 0$ with r held fixed, which describes the situation far from the centerline $\mathcal{C}$ and an inner problem defined by the *inner limit* $\epsilon \to 0$ with $\bar{r} = r/\epsilon$ held fixed, which describes the situation near this centerline. The *inner expansion* $\mathbf{f}^{\,\text{inn}}(\mathbf{x}, \epsilon)$ of a vector field $\mathbf{f}(\mathbf{x}, \epsilon)$ is the expansion $\epsilon \to 0$ of $\mathbf{f}(\mathbf{x}, \epsilon)$ in terms of $\bar{r} = r/\epsilon$ held fixed and the *outer expansion* $\mathbf{f}^{\,\text{out}}(\mathbf{x}, \epsilon)$ is the expansion $\epsilon \to 0$ of $\mathbf{f}(\mathbf{x}, \epsilon)$ with r held fixed. The velocity induced by this vortex is given by the Biot-Savart law

$$\mathbf{v}(\mathbf{x}) = \frac{1}{4\pi} \iiint \frac{\boldsymbol{\omega}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} \mathrm{d}\mathbf{x}', \tag{18}$$

which can be written in local coordinates near the centerline $\mathcal{C}$

$$\mathbf{v}(r, \varphi, s, t, \epsilon) = \frac{1}{4\pi} \iiint \frac{\epsilon^2 \boldsymbol{\omega}(\bar{r}', \varphi', a', \epsilon) \times \left[\mathbf{X} + r\mathbf{e}_r - (\mathbf{X}' + \epsilon \bar{r}' \mathbf{e}_r')\right]}{\left|\mathbf{X} + r\mathbf{e}_r - (\mathbf{X}' + \epsilon \bar{r}' \mathbf{e}_r')\right|^3} h_3' \bar{r}' \mathrm{d}\bar{r}' \mathrm{d}\varphi' \mathrm{d}a', \tag{19}$$

where $h_3' = \big(1 - K(a')\epsilon \bar{r}' \cos(\varphi')\big)$. In this section, since the parameter ϵ is small, we want to find the expansion of Equation (19) in terms of ϵ. Thus, for the given vorticity field $\boldsymbol{\omega}(\mathbf{x}) = \boldsymbol{\omega}(\bar{r}, \varphi, a, \epsilon) = \epsilon^{-2} \boldsymbol{\omega}^{(0)}(\bar{r}, \varphi, a)$, we are seeking the following expansions

$$\begin{array}{llllll} \boldsymbol{\omega}^{\text{out}} & = & \boldsymbol{\omega}^{\text{out}(0)} & +\epsilon & \boldsymbol{\omega}^{\text{out}(1)} & +O(\epsilon^2), \\ \mathbf{v}^{\text{out}} & = & \mathbf{v}^{\text{out}(0)} & +\epsilon & \mathbf{v}^{\text{out}(1)} & +O(\epsilon^2), \\ \mathbf{v}^{\text{inn}} & = \epsilon^{-1} & \mathbf{v}^{\text{inn}(0)} & + & \mathbf{v}^{\text{inn}(1)} & +O(\epsilon). \end{array} \tag{20}$$

We denote the radial, circumferential and axial components of the vorticity field $\boldsymbol{\omega}$ by ω_1, ω_2, and ω_3, respectively, *i.e.*,

$$\boldsymbol{\omega} = \omega_1 \mathbf{e}_r + \omega_2 \mathbf{e}_\varphi + \omega_3 \mathbf{t}.$$

The conservation law $\mathrm{div}\boldsymbol{\omega} = 0$ in the curvilinear coordinates is

$$(\omega_1 r h_3)_r + (h_3 \omega_2)_\varphi + r\omega_{3a} - Tr\omega_{3\varphi} = 0, \tag{21}$$

where T is the torsion of the centerline. The conservation of the circulation along the vortex also gives $\iint \omega_3 r \mathrm{d}r \mathrm{d}\varphi = 1$.

We first perform the *outer expansion* of the Biot-Savart law (19) and obtain

$$\mathbf{v}^{\text{out}(0)}(\mathbf{x}) = \frac{1}{4\pi} \iiint \frac{\boldsymbol{\omega}^{(0)}(\bar{r}', \varphi', a') \times \big(\mathbf{x} - \mathbf{X}(a')\big)}{|\mathbf{x} - \mathbf{X}(a')|^3} \bar{r}' \mathrm{d}\bar{r}' \mathrm{d}\varphi' \mathrm{d}a', \tag{22}$$

where $\mathbf{x} = \mathbf{X}(a) + r\mathbf{e}_r(\varphi, a)$. In fact, this expression can be simplified, and to do so, let us define

$$\mathbf{D}(a) \equiv \iint \boldsymbol{\omega} r \mathrm{d}r \mathrm{d}\varphi - \mathbf{t} = \iint (\boldsymbol{\omega} - \omega_3 \mathbf{t})\, r \mathrm{d}r \mathrm{d}\varphi = \iint \big(\omega_1 \mathbf{e}_r + \omega_2 \mathbf{e}_\varphi\big)\, r \mathrm{d}r \mathrm{d}\varphi.$$

The definitions of $\mathbf{e}_r$ and $\mathbf{e}_\varphi$

$$\mathbf{e}_r = \cos\varphi \mathbf{n} + \sin\varphi \mathbf{b}, \qquad \mathbf{e}_\varphi = -\sin\varphi \mathbf{n} + \cos\varphi \mathbf{b},$$

and an integration by parts gives

$$\begin{aligned}\mathbf{D}(a) &= \mathbf{n}\iint (\omega_1\cos\varphi - \omega_2\sin\varphi)\, r\mathrm{d}r\mathrm{d}\varphi + \mathbf{b}\iint (\omega_1\sin\varphi + \omega_2\cos\varphi)\, r\mathrm{d}r\mathrm{d}\varphi \\ &= \mathbf{n}\iint (\omega_1\cos\varphi - \omega_2\sin\varphi)\, r\mathrm{d}r\mathrm{d}\varphi + \mathbf{b}\iint \big(\omega_1\sin\varphi - (\omega_2)_\varphi\sin\varphi\big)\, r\mathrm{d}r\mathrm{d}\varphi.\end{aligned}$$

For the vorticity field $\boldsymbol{\omega}(\mathbf{x}) = \epsilon^{-2}\boldsymbol{\omega}^{(0)}(\bar{r},\varphi,a)$, Equation (21) gives

$$(\omega_1\bar{r})_{\bar{r}} + (\omega_2)_\varphi = 0, \qquad -K(a)\,[\omega_1\cos\varphi - \omega_2\sin\varphi] + \omega_{3a} - T(a)\omega_{3\varphi} = 0.$$

We use these two equations to find that

$$\begin{aligned}\mathbf{D}(a) &= \frac{\mathbf{n}}{K}\iint \omega_{3a} r\mathrm{d}r\mathrm{d}\varphi + \mathbf{b}\iint (\omega_1\sin\varphi + (\omega_1\bar{r})_{\bar{r}}\sin\varphi)\, r\mathrm{d}r\mathrm{d}\varphi \\ &= \mathbf{b}\iint (\omega_1 + (\omega_1\bar{r})_{\bar{r}})\sin\varphi r\mathrm{d}r\mathrm{d}\varphi = \mathbf{b}\iint (\epsilon^2\omega_1\bar{r}^2)_{\bar{r}}\sin\varphi\mathrm{d}\bar{r}\mathrm{d}\varphi = 0,\end{aligned}$$

where we finally have used the fact that $\omega_1\bar{r}^2 = 0$ at infinity. So $\mathbf{D}(a) = 0$, *i.e.* $\iint \boldsymbol{\omega} r\mathrm{d}r\mathrm{d}\varphi = \mathbf{t}$, and Equation (22) simplifies to yield

$$\mathbf{v}^{\mathrm{out}(0)}(r,\varphi,a) = \frac{1}{4\pi}\int_{\mathcal{C}} \frac{\mathbf{t}(a')\times\big(\mathbf{x}-\mathbf{X}(a')\big)}{|\mathbf{x}-\mathbf{X}(a')|^3}\mathrm{d}a'. \tag{23}$$

This shows that, at leading order, the outer velocity field exactly corresponds to the Biot-Savart law applied to the Dirac delta distribution $\boldsymbol{\omega}^{\mathrm{out}(0)} = \delta_{\mathcal{C}}\mathbf{t}$ on the centerline $\mathcal{C}$. The next order $\mathbf{v}^{\mathrm{out}(1)}(r,\varphi,a)$ is indeed not zero and is given in Margerit [14].

We now perform the *inner expansion* of the Biot-Savart law (19) and to do so we first introduce the stretched inner variable $\bar{r} = r/\epsilon$ in this integral (19) which becomes

$$\mathbf{v}(\bar{r},\varphi,a,\epsilon) = \frac{1}{4\pi}\iiint \frac{\epsilon^2\boldsymbol{\omega}(\bar{r}',\varphi',a',\epsilon)\times\big[\mathbf{X}+\epsilon\bar{r}\mathbf{e}_r - (\mathbf{X}'+\epsilon\bar{r}'\mathbf{e}'_r)\big]}{\big|\mathbf{X}+\epsilon\bar{r}\mathbf{e}_r - (\mathbf{X}'+\epsilon\bar{r}'\mathbf{e}'_r)\big|^3} h'_3\bar{r}'\mathrm{d}\bar{r}'\mathrm{d}\varphi'\mathrm{d}a'. \tag{24}$$

This integral (24) is a singular integral in terms of the small parameter ϵ. In order to find its expansion in terms of ϵ, we use the matched asymptotic expansion of singular integrals method (MAESI) which has been described in Section 2 for the simpler case of a line vortex. This method consists in splitting the integration in the axial direction a' into two parts. In an *outer region* outside a neighbourhood of the points $\mathbf{M}(\bar{r}',\varphi',a)$ the integrand is expanded in terms of ϵ with $a^* = a' - a$ held fixed and then integrated. In an *inner region* in a neighbourhood of the points $\mathbf{M}(\bar{r}',\varphi',a)$ the stretched inner variable $\bar{a} = a^*/\epsilon$ is introduced. The found integrand is expanded in terms of ϵ with the stretched inner variable held fixed and then integrated. The last step is the asymptotic matching which consists in adding these two integrated expansions. A straightforward calculation gives

$$\mathbf{v}^{\mathrm{inn}(0)} = -\frac{1}{2\pi}\iint \mathbf{g}\bar{r}'\mathrm{d}\bar{r}'\mathrm{d}\varphi', \tag{25}$$

where

$$\begin{aligned}\mathbf{g} &= \boldsymbol{\omega}^{(0)}(\bar{r}',\varphi',a)\times\big[\bar{r}'\mathbf{e}_r(\varphi',a) - \bar{r}\mathbf{e}_r(\varphi,a)\big]/k^2, \\ k^2 &= \bar{r}^2 + \bar{r}'^2 - 2\bar{r}\bar{r}'\cos(\varphi-\varphi').\end{aligned}$$

At first order we have

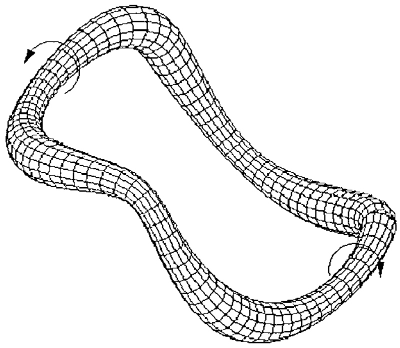

Figure 2. The domain of non-zero vorticity.

$$\mathbf{v}^{\text{inn}(1)} = \mathbf{A} + \frac{K}{4\pi}[\log\frac{S}{\epsilon} - 1]\mathbf{b} - \frac{1}{4\pi}\iint \boldsymbol{\omega}_a^{(0)}(\bar{r}', \varphi', a) \times \mathbf{t}(a)\log\frac{1}{k^2}\bar{r}'\mathrm{d}\bar{r}'\mathrm{d}\varphi'$$

$$-\frac{K(a)}{8\pi}\iint \boldsymbol{\omega}^{(0)}(\bar{r}', \varphi', a) \times \mathbf{n}(a)\log\frac{1}{k^2}\bar{r}'\mathrm{d}\bar{r}'\mathrm{d}\varphi' - \frac{K(a)\bar{r}\cos(\varphi)}{4\pi}\iint \mathbf{g}\bar{r}'\mathrm{d}\bar{r}'\mathrm{d}\varphi' \qquad (26)$$

$$-\frac{T(a)\bar{r}}{2\pi}\iint \frac{\boldsymbol{\omega}^{(0)}(\bar{r}', \varphi', a) \times \mathbf{t}(a)}{k^2}\sin(\varphi - \varphi')\bar{r}'^2\mathrm{d}\bar{r}'\mathrm{d}\varphi' + \frac{K(a)}{4\pi}\iint \mathbf{g}\cos\varphi'\bar{r}'^2\mathrm{d}\bar{r}'\mathrm{d}\varphi',$$

where $\mathbf{A}$ is given by (10). These two terms (25) and (26) give the velocity field near and within the vortex ring of vorticity $\boldsymbol{\omega} = \epsilon^{-2}\boldsymbol{\omega}^{(0)}$. If the vorticity has only a tangential component, the term (25) of order $1/\epsilon$ is the two-dimensional Biot-Savart law. This term (25) of order $1/\epsilon$ and the term of order $\log\epsilon$ in (26) were initially found by Levi-Civita [15–17]. The term of order 1 in (26) was given on the centerline $\bar{r} = 0$ for an axisymmetric vorticity by Klein and Knio [18]. The expansion $\mathbf{v}^{\text{inn}}(\bar{r} \to \infty, \varphi, a, \epsilon)$ can be found with the help of the above expansion of $\mathbf{v}^{\text{inn}}$. The matching law states that the substitution $\bar{r} = r/\epsilon$ in the limit $\mathbf{v}^{\text{inn}}(\bar{r} \to \infty, \varphi, a, \epsilon)$ gives the limit $\mathbf{v}^{\text{out}}(r \to 0, \varphi, a)$, *i.e.* expression (14). That has been effectively checked up to order $O(r)$ by use of the expression of $\mathbf{v}^{\text{inn}(2)}$ obtained with the help of a computer-algebra system (Maple).

5. The inner expansion of the Biot-Savart law applied to a family of slender vortices with axial core variation

In this section, the previous expansion (25–26) of the Biot-Savart law is used to obtain the inner expansion of the velocity field for the following family of slender vortex rings with axial core variation (Figure 2), namely

$$\boldsymbol{\omega} = \frac{1}{\epsilon^2}\left[\frac{1}{\pi\bar{r}_0^2(a)}\mathbf{t} + \epsilon\frac{\bar{r}_0'(a)}{\pi\bar{r}_0^3(a)}\frac{\bar{r}}{h_3}\mathbf{e}_r + \frac{g(a,\bar{r})}{h_3}\mathbf{e}_\varphi\right]H\left(1 - \frac{\bar{r}}{\bar{r}_0(a)}\right), \qquad (27)$$

where $\bar{r}_0(a)$ is the core radius of a slender vortex ring of the general curved centerline $\mathbf{X}(a)$; $\bar{r}_0'(a)$ is the axial derivative $\mathrm{d}\bar{r}_0(a)/\mathrm{d}a$, H is the Heaviside function, and $g(a, \bar{r})$ is an arbitrary function. We constructed this family (27) of vortex rings with axial variation as an exact solution of the conservation law div$\boldsymbol{\omega} = 0$ – which can be checked with the help of Equation (21) – and of the normal condition $\boldsymbol{\omega}\cdot\mathbf{N} = 0$ on the interface $\bar{r} = \bar{r}_0(a)$, where $\mathbf{N}$ is the normal to this interface which is proportional to $\mathbf{e}_r - \bar{r}_0'/h_3\mathbf{t}$. In the following we will consider vortex rings without circumferential component ($g(a, \bar{r}) = 0$) of the vorticity field.

The leading-order *outer expansion* of the induced velocity is the velocity (23) induced by a line vortex on the centerline $\mathbf{X}(a)$. In order to analyse the velocity field induced in the inner region, the relative velocity $\mathbf{V}$ is defined by $\mathbf{v} = \mathbf{v}(\bar{r} = 0, a) + \mathbf{V}$, where $\mathbf{v}(\bar{r} = 0, a)$ is the velocity field on the centerline. We denote the radial, circumferential and axial components of the relative velocity field $\mathbf{V}$ by u, v, w, respectively, *i.e.*, $\mathbf{V} = u\mathbf{e}_r + v\mathbf{e}_\varphi + w\mathbf{t}$. The inner expansion of $\mathbf{V}$ is taken to be of the form $\mathbf{V}^{\text{inn}} = \epsilon^{-1}\mathbf{V}^{\text{inn}(0)} + \mathbf{V}^{\text{inn}(1)} + \ldots$. The straightforward use of the inner expansion of the Biot-Savart law (25–26) gives the *inner expansion* of the induced velocity

$$\mathbf{V}^{\text{inn}(0)} = \begin{cases} \dfrac{\bar{r}^*}{2\pi\bar{r}_0}\mathbf{e}_\varphi & \text{if } \bar{r}^* < 1, \\[2ex] \dfrac{1}{2\pi\bar{r}^*\bar{r}_0}\mathbf{e}_\varphi & \text{if } \bar{r}^* > 1, \end{cases} \tag{28}$$

$$\mathbf{V}^{\text{inn}(1)} = \begin{cases} -\bar{r}^{*2}\dfrac{K}{16\pi}\left[3\sin\varphi\mathbf{e}_r + \cos\varphi\mathbf{e}_\varphi\right] & \text{if } \bar{r}^* < 1, \\[2ex] -\dfrac{K}{16\pi}\left(\left[-\dfrac{1}{\bar{r}^{*2}} + 4 + 4\log\bar{r}^*\right]\sin\varphi\mathbf{e}_r \right. & \\ \qquad\left. + \left[\dfrac{1}{\bar{r}^{*2}} + 4\log\bar{r}^*\right]\cos\varphi\mathbf{e}_\varphi\right) & \text{if } \bar{r}^* > 1, \end{cases} \tag{29}$$

and

$$\mathbf{v}(\bar{r} = 0, a) = \mathbf{A} + \frac{K}{4\pi}\left[\log\frac{S}{\epsilon\bar{r}_0}\right]\mathbf{b}, \tag{30}$$

where $\bar{r}^* = \bar{r}/\bar{r}_0$ and the global integral $\mathbf{A}$ is given by (10). In order to obtain this result (28–30), we have used the expressions of the following integrals given by Gradshteyn [19, pp. 409, pp. 621–622]

$$\int_0^{2\pi} \frac{-1 + a\cos x}{(1 - 2a\cos x + a^2)}\mathrm{d}x = \begin{cases} -2\pi & \text{if } a < 1, \\ 0 & \text{if } a > 1, \end{cases}$$

$$\int_0^{2\pi} \log(1 - 2a\cos x + a^2)\mathrm{d}x = \begin{cases} 0 & \text{if } a < 1, \\ 2\pi\log a^2 & \text{if } a > 1, \end{cases}$$

$$\int_0^{2\pi} \frac{\cos nx}{(1 - 2a\cos x + a^2)}\mathrm{d}x = \begin{cases} 2\dfrac{\pi a^n}{1 - a^2} & \text{if } a < 1, \\[2ex] 2\dfrac{\pi}{a^n(a^2 - 1)} & \text{if } a > 1, \end{cases}$$

$$\int_0^{2\pi} \frac{\sin nx \sin x}{(1 - 2a\cos x + a^2)}\mathrm{d}x = \begin{cases} \pi a^{n-1} & \text{if } a < 1, \\[1ex] \dfrac{\pi}{a^{n+1}} & \text{if } a > 1, \end{cases}$$

$$\int_0^{2\pi} \frac{-1 + a\cos x}{(1 - 2a\cos x + a^2)}\cos x\mathrm{d}x = \begin{cases} -\pi a & \text{if } a < 1, \\[1ex] \dfrac{\pi}{a} & \text{if } a > 1. \end{cases}$$

The velocity up to order 1 is given in the inner region by (28–30) and in the outer region by (23). The axial derivative $\bar{r}_0'$ of the core radius and the local torsion T of the centerline do not appear in this inner expansion of the velocity field up to this order. In order to check our result, we apply the curl in local coordinates to the obtained velocity field (28–30). The vorticity we derived agrees with the two first orders of the vorticity (27). The next order of the vorticity field depends on the order $O(\epsilon)$ velocity field. The velocity (30) on the centerline depends only of the axial coordinates a and does not affect these two first orders of the vorticity field. We checked also that the two first orders of the continuity equation are satisfied. The velocity (30) does not affect these two first orders of the continuity equation. To check the velocity (30) on the centerline, we use the matching law and compare the behaviour of the order $O(1)$ velocity field (28–30) at infinity with the one of the outer velocity field near the line at order $O(1)$ in terms of r given by (14). This comparison confirms the expression (30) of the velocity field on the centerline. If the centerline is a circle, the global integral $\mathbf{A}$ is $\mathbf{A} = K\log(8/2\pi)\mathbf{b}/(4\pi)$, as one can deduce from a comparison between the dimensionless velocity of a circular vortex ring $K\left[\log[8S/(2\pi\epsilon)] + C_v(t) - 1 + C_w(t)\right]/(4\pi)$ and the equation of motion (15) of Callegari and Ting.

As in Callegari and Ting [5] a stream function $\psi^{(1)}$ with

$$u^{(1)} = \frac{1}{\bar{r}}\psi_{\varphi}^{(1)} \quad \text{and} \quad v^{(1)} = -\psi_{\bar{r}}^{(1)} + \bar{r}v^{(0)}K\cos(\varphi)$$

describes the order-one velocity. This stream function is found from (28–29) and is in the form $\psi^{(1)} = \tilde{\psi}_{11}\cos(\varphi)$ with

$$\tilde{\psi}_{11} = \begin{cases} \dfrac{3\bar{r}_0 K}{16\pi}\bar{r}^{*3} & \text{if } \bar{r}^* < 1, \\[2ex] \dfrac{\bar{r}_0 K}{16\pi}\left[-\dfrac{1}{\bar{r}^*} + 4\bar{r}^*\left(1 + \log\bar{r}^*\right)\right] & \text{if } \bar{r}^* > 1, \end{cases} \tag{31}$$

and $\bar{r}^* = \bar{r}/\bar{r}_0$. The streamlines and the velocity field associated with this stream function $\psi^{(1)}$ are displayed in Figure 3. This figure gives a geometrical description of the matching law and of the binormal component of the velocity on the centerline.

The family of vorticity fields (27) and its induced velocity (28–30) give an example of a three-dimensional slender vortex ring with axial core variation that may be an initial condition to Navier-Stokes equations. We may raise the question of the time evolution of this core variation on the vortex ring and of the motion of its centerline. Even when $\bar{r}_0$ is a constant, the velocity (30) of the centerline does not correspond to the single-time-scale asymptotic solution (15) found by Callegari and Ting [5]. In this single-time analysis the parameters C_v and C_w in Equation (15) are $C_v = 3/4 - \log\bar{r}_0$ and $C_w = 0$ for a leading-order velocity (28). The velocity on the centerline found from Equation (15) is then

$$\mathbf{v}(\bar{r} = 0, a) = \mathbf{A} + \frac{K}{4\pi}\left[-\frac{1}{4} + \log\frac{S}{\epsilon\bar{r}_0}\right]\mathbf{b} \tag{32}$$

and the stream function $\tilde{\psi}_{11}$ is

$$\tilde{\psi}_{11} = \begin{cases} \dfrac{5\bar{r}_0 K}{16\pi}\bar{r}^{*3} & \text{if } \bar{r}^* < 1, \\[2ex] \dfrac{\bar{r}_0 K}{16\pi}\left[\dfrac{2}{\bar{r}^*} + 4\bar{r}^*\left(\dfrac{3}{4} + \log\bar{r}^*\right)\right] & \text{if } \bar{r}^* > 1, \end{cases} \tag{33}$$

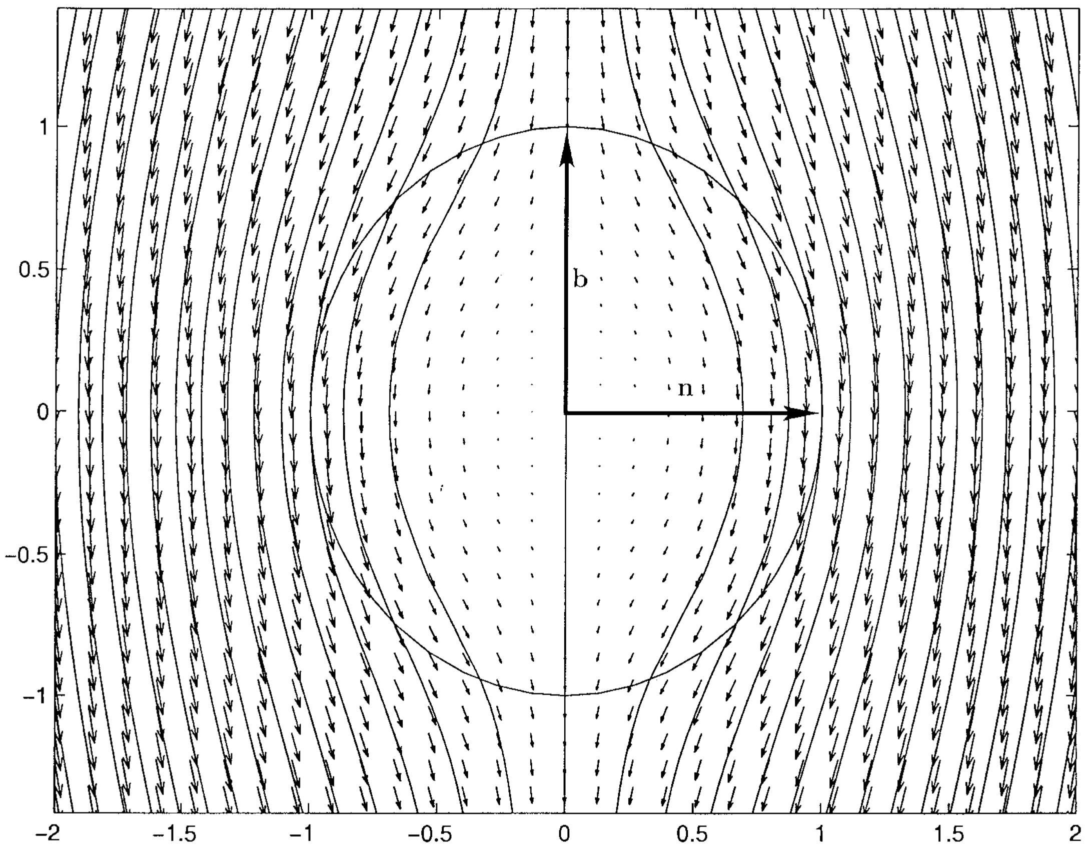

Figure 3. The streamlines and the velocity field associated with the stream function $\psi^{(1)}$ of Equation (31). The circle is the interface $\bar{r}^* = 1$.

where $\bar{r}^* = \bar{r}/\bar{r}_0$. The two first orders of vorticity are

$$\boldsymbol{\omega} = \frac{1}{\epsilon^2}\left[\left(\frac{1}{\pi\bar{r}_0^2} - \frac{K\bar{r}}{\pi\bar{r}_0^2}\cos(\varphi)\epsilon + O(\epsilon^2)\right)\mathbf{t}\right] H\left(1 - \frac{\bar{r}}{\bar{r}_0}\right). \tag{34}$$

The difference between the order $O(1/\epsilon)$ tangential vorticity of the single-time-scale analysis (34) and the one (27) of the family studied here explains the difference between the velocities on the centerline (30) and (32). It also explains the difference (Figure 4) between the stream functions (31) and (33). The vortex ring (27) without axial variation is an example of a three-dimensional initial condition of the Navier-Stokes equation for which the study of the time evolution would require a double-time-scale expansion with a normal time t and a fast time $\bar{t} = t/\epsilon^2$ as was introduced in two dimensions by Ting and Tung [20, 21].

We can add an axial flow of strength m to this flow (28–30) by adding the following field of velocity

$$\mathbf{v} = \frac{1}{\epsilon}\left[\frac{m}{\pi\bar{r}_0^2(a)}\mathbf{t} + \epsilon m\frac{\bar{r}_0'(a)}{\pi\bar{r}_0^3(a)}\frac{\bar{r}}{h_3}\mathbf{e}_r + \frac{f(a,\bar{r})}{h_3}\mathbf{e}_\varphi\right] H\left(1 - \frac{\bar{r}}{\bar{r}_0(a)}\right), \tag{35}$$

where $f(a, \bar{r})$ is an arbitrary function. We constructed this field as an exact solution of the continuity equation $\mathrm{div}\mathbf{v} = 0$ and of the normal condition $\mathbf{v}\cdot\mathbf{N} = 0$ on the interface $\bar{r} = \bar{r}_0(a)$ of the vortex ring. At leading order this field (35) is a vortex sheet $\boldsymbol{\omega}(0) = m\delta_{\bar{r}=\bar{r}_0}/\bar{r}_0^2\mathbf{e}_\varphi$. When $f(a, \bar{r}) = 0$, the first orders of this velocity field are

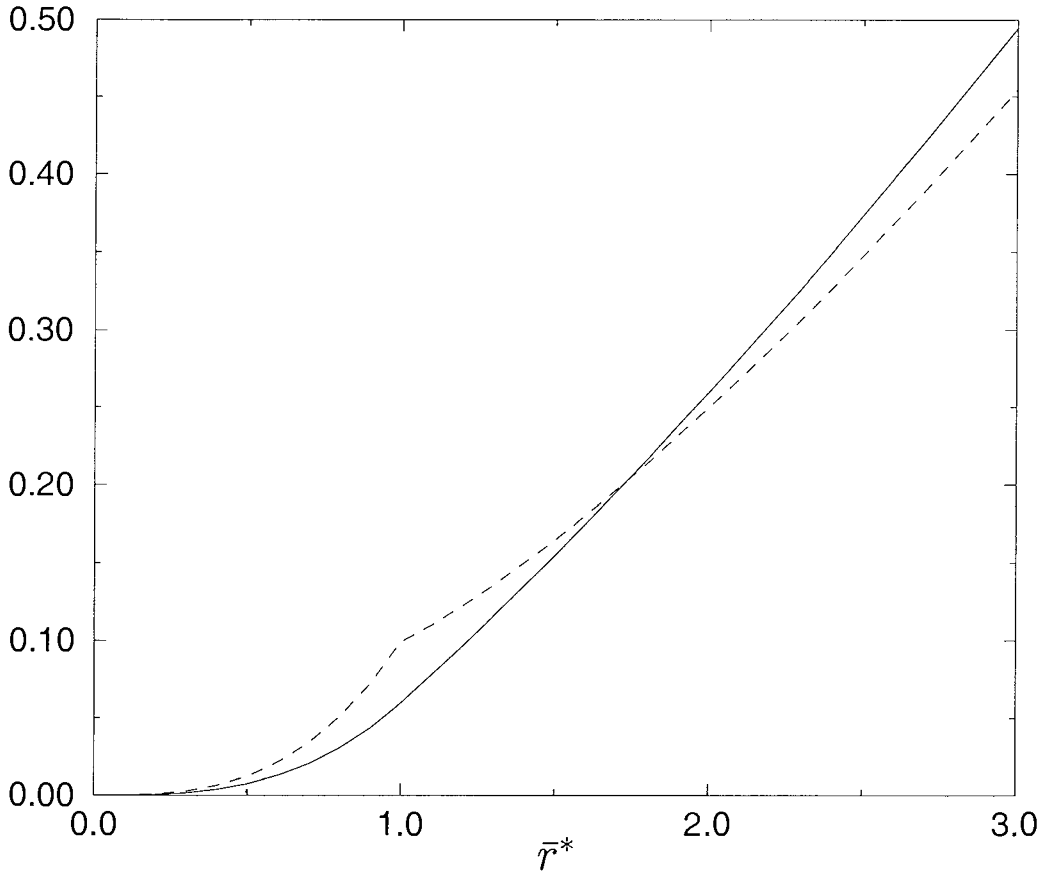

Figure 4. Radial evolution of the stream function $\tilde{\psi}_{11}$. The solid line is from Equation (31) and the dashed line is from Equation (33).

$$\mathbf{V}^{\mathrm{inn}(0)} = \begin{cases} \dfrac{m}{\pi \bar{r}_0^2}\mathbf{t} & \text{if } \bar{r}^* < 1, \\ 0 & \text{if } \bar{r}^* > 1, \end{cases} \tag{36}$$

$$\mathbf{V}^{\mathrm{inn}(1)} = \begin{cases} \dfrac{m\bar{r}_0'(a)}{\pi \bar{r}_0^2}\bar{r}^*\mathbf{e}_r & \text{if } \bar{r}^* < 1, \\ 0 & \text{if } \bar{r}^* > 1. \end{cases} \tag{37}$$

If these velocities (36–37) are added to (28) and (29), this adds an axial flow of strength m to the velocity field of the vortex ring previously studied.

6. The leading-order dynamical equations of the axisymmetric axial core variation on a curved slender vortex

In this section we investigate the time evolution of axisymmetric axial core variation on vortex filament in order to understand the time evolution of the initial conditions (27–30) to the

Navier-Stokes equations. We denote the radial, circumferential and axial components of the relative velocity field $\mathbf{V}$ by u, v, w, *i.e.*, $\mathbf{V} = u\mathbf{e}_r + v\mathbf{e}_\varphi + w\mathbf{t}$. In a single-time-scale analysis for a vortex ring with axial core variation the inner expansions of the relative velocity components are

$$\begin{aligned} u^{\text{inn}} &= & u^{(1)}(\bar{r}, \varphi, s, t) \ + \ldots, \\ v^{\text{inn}} &= \epsilon^{-1} v^{(0)}(\bar{r}, s, t) \ + & v^{(1)}(\bar{r}, \varphi, s, t) \ + \ldots, \\ w^{\text{inn}} &= \epsilon^{-1} w^{(0)}(\bar{r}, s, t) \ + & w^{(1)}(\bar{r}, \varphi, s, t) \ + \ldots. \end{aligned}$$

Unfortunately, for the initial conditions (27–30) one cannot just use the single-time-scale analysis for a vortex ring with axial core variation as given by Klein and Ting [22], because the symmetric part of $\mathbf{V}^{\text{inn}(1)}$ in Equation (29) does not satisfy the following compatibility conditions of the single-time-scale analysis for a vortex ring with axial core variation given by Ting and Klein [5, 22]

$$\begin{aligned} &(\bar{r}u_c^{(1)})_{\bar{r}} + \frac{\bar{r}}{\sigma^{(0)}} w_s^{(0)} \ = \ 0, \qquad \frac{\left(\bar{r}v^{(0)}\right)_{\bar{r}}}{\bar{r}} u_c^{(1)} + \frac{w^{(0)}}{\sigma^{(0)}} v_s^{(0)} = 0, \\ &w_{\bar{r}}^{(0)} u_c^{(1)} + \frac{p_s^{(0)}}{\sigma^{(0)}} + \frac{w^{(0)}}{\sigma^{(0)}} w_s^{(0)} \ = \ 0, \qquad p^{(0)} = -\int_{\bar{r}}^{\infty} \frac{v^{(0)2}}{\bar{r}} \mathrm{d}\bar{r}, \end{aligned} \tag{38}$$

where $u_c^{(1)}$ is the axisymmetric part of the radial velocity at order unity and $p^{(0)}$ is the leading-order pressure. Equations (28–29) give $v^{(0)} \neq 0$, $w^{(0)} = 0$, and $u_c^{(1)} = 0$. The third equation of (38) is then not satisfied, as the axial derivative $p_s^{(0)}$ of the pressure is not equal to zero. The vortex ring (27) without axial variation is the only one that satisfies these compatibility conditions.

In fact, the time of evolution for these initial conditions (27–30) and for non-short axial core variation on a filament is a time $\tau = t/\epsilon$ that is in-between the time $\bar{t} = t/\epsilon^2$ of evolution for a non-axisymmetric core and the time t of motion for a curved vortex. This regime is not the same as the one considered by Ting and Klein [23, pp. 181–185] , who studied axial core variation on an open vortex filament by means of a single-time-scale t and double-axial-scale $(s, \xi = \epsilon s)$ analysis. In the double-time-scale analysis $(t, \tau = t/\epsilon)$ for an open filament the long-time t behaviour of a core variation perturbation that evolves at short time $\tau = t/\epsilon$ is to reach the far distance $\xi = \epsilon s$ of the Ting and Klein [23] regime. In the following the leading-order equations of this double-time-scale analysis are given.

With this double-time-scale analysis the inner expansions of the relative velocity components are

$$\begin{aligned} u^{\text{inn}} &= & u^{(1)}(\bar{r}, \varphi, s, t, \tau) \ + \ldots, \\ v^{\text{inn}} &= \epsilon^{-1} v^{(0)}(\bar{r}, s, t, \tau) \ + & v^{(1)}(\bar{r}, \varphi, s, t, \tau) \ + \ldots, \\ w^{\text{inn}} &= \epsilon^{-1} w^{(0)}(\bar{r}, s, t, \tau) \ + & w^{(1)}(\bar{r}, \varphi, s, t, \tau) \ + \ldots. \end{aligned}$$

For this small time the derivatives with the small-time τ appear in the compatibility conditions (38) of the single-time-scale analysis and these Equations become the following equations of evolution for the axisymmetric part of the relative velocity field

$$\begin{aligned} &(\bar{r}u_c^{(1)})_{\bar{r}} + \frac{\bar{r}}{\sigma^{(0)}} w_s^{(0)} = \ 0, \qquad \frac{\partial v^{(0)}}{\partial \tau} + \frac{\left(\bar{r}v^{(0)}\right)_{\bar{r}}}{\bar{r}} u_c^{(1)} + \frac{w^{(0)}}{\sigma^{(0)}} v_s^{(0)} = 0, \\ &\frac{\partial w^{(0)}}{\partial \tau} + w_{\bar{r}}^{(0)} u_c^{(1)} + \frac{p_s^{(0)}}{\sigma^{(0)}} + \frac{w^{(0)}}{\sigma^{(0)}} w_s^{(0)} = \ 0, \qquad p^{(0)} = -\int_{\bar{r}}^{\infty} \frac{v^{(0)2}}{\bar{r}} \mathrm{d}\bar{r}. \end{aligned} \tag{39}$$

When $\sigma = 1$, the parameter on the centerline is an arclength and these equations are the 'long-wave scaling' shallow-water equations derived from studies of vortex breakdown of a straight filament [24]. Similar shallow-water equations have been deduced in an ad-hoc way by Lundgren and Ashurts [11].

Let us define the meridional stream function ψ, with

$$u_c^{(1)} = -\frac{1}{\sigma^{(0)}\bar{r}}\psi_s, \qquad w^{(0)} = \frac{1}{\bar{r}}\psi_{\bar{r}},$$

and introduce the following transformation [24]

$$\mathcal{G} = \bar{r}v^{(0)}, \qquad y = \bar{r}^2.$$

In these new variables the system (39) becomes

$$\begin{aligned} \frac{\partial \mathcal{G}}{\partial \tau} - \frac{2}{\sigma^{(0)}}\psi_s \mathcal{G}_y + \frac{2}{\sigma^{(0)}}\psi_y \mathcal{G}_s &= 0, \\ D^2\frac{\partial \psi}{\partial \tau} + \frac{2}{\sigma^{(0)}}\psi_y D^2\psi_s + \frac{2}{y\sigma^{(0)}}\mathcal{G}\mathcal{G}_s - \frac{2}{\sigma^{(0)}}y\psi_s\left[y^{-1}D^2\psi\right]_y &= 0, \end{aligned} \tag{40}$$

where $D^2\psi = rw_r^{(0)} = 4y\psi_{yy}$. The axisymmetric parts of the velocity fields (28–29) and (36–37) give initial conditions to these equations of evolution on the small-time τ and their numerical integration is currently under investigation.

7. Conclusion

We have used the method of Matched Asymptotic Expansion of Singular Integrals (MAESI) to obtain the inner expansion of the Biot-Savart law for a slender vortex with core variation. This expansion has been carried out in terms of the thickness of the filament and is the first inner expansion up to $O(1)$ of the Biot-Savart law for slender vortex filaments.

The MAESI method has been previously applied to the simpler case of the known expansion, in terms of the small-distance r to a line vortex, of the potential flow (3) induced by this line. This derivation is an alternative to the technique of the osculating circle initially used by Widnall *et al.* The successive steps of this derivation have been displayed so as to describe how this method works. This also provides an example of the expansion of a singular integral in terms of a small parameter and this example may be useful for other expansions of the same kind in fluid dynamics (*e.g.* expansion of ad-hoc de-singularized integrals) but also in other fields (*e.g.* electromagnetics). The relation between the cut-off length introduced in the cut-off line-integral technique and the inner-core parameters C_v and C_w defined by Callegari and Ting has also been given.

The obtained inner expansion of the Biot-Savart law was finally used to give the inner expansion of the velocity field induced by a family of curved vortex rings with axial-core variation. This expansion was given up to order one in terms of the thickness ϵ of these vortices. This family of vorticity fields gives an interesting example of initial conditions for the Navier-Stokes equations. In order to understand the time-evolution of these initial conditions, a short-time scale was introduced. This time is in-between the time of the evolution of a non-axisymmetric core and the time of motion of a curved vortex. A quasi-hyperbolic system that describes the leading-order dynamics of axisymmetric axial-core variation on a curved slender vortex filament was finally extracted from the Navier-Stokes equations and is of interest for comparison with systems obtained in an ad-hoc way such as the one proposed by Lundgren and Ashurst.

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